

This discussion is inspired in the works of Bogoslovsky [1] and Dhasmana and Silagadze [2]

1 Review of Lorentz Transformations

We assume the space-time homogeneous. It means that the transformations between two systems of coordinates (only two dimensions by simplicity) S' and S , with relative velocity v are linears:

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} \alpha(v) & \sigma(v) \\ \delta(v) & \lambda(v) \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix} \quad (1)$$

It can only depend on the relative velocity.

We consider both observers start together at $x = x' = 0$. Also, we consider S' is in rest and after a time t for S we have

$$\begin{pmatrix} ct' \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha & \sigma \\ \delta & \lambda \end{pmatrix} \begin{pmatrix} ct \\ vt \end{pmatrix} \quad (2)$$

From the second component we have

$$\delta = -\lambda\beta \quad (3)$$

with $\beta = \frac{v}{c}$.

Hence,

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} \alpha(v) & \sigma(v) \\ -\lambda(v)\beta & \lambda(v) \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix} \quad (4)$$

From it, we have for the spatial part

$$x' = \lambda(v)(x - vt) \quad (5)$$

The inverse transformation should be the same with the opposite direction in the velocity $(-v)$:

$$\begin{pmatrix} ct \\ x \end{pmatrix} = \begin{pmatrix} \alpha(-v) & \sigma(-v) \\ \lambda(-v)\beta & \lambda(-v) \end{pmatrix} \begin{pmatrix} ct' \\ x' \end{pmatrix} \quad (6)$$

For the spatial part:

$$x = \lambda(-v)(x' + vt') \quad (7)$$

We put together both spatial parts:

$$x' = \lambda(v)x - \lambda(v)\beta ct \quad (8)$$

$$x = \lambda(-v)x' + \lambda(-v)\beta ct' \quad (9)$$

Inserting x' from (8) in the equation (9) and arranging:

$$ct' = \left(\frac{1}{\lambda(-v)} - \lambda(v) \right) \frac{1}{\beta} x + \lambda(v) ct \quad (10)$$

From eq. (4) we have for the temporal part:

$$ct' = \alpha(v) ct + \sigma(v) x \quad (11)$$

equating (10) and (11):

$$\alpha(v) = \lambda(v) \quad (12)$$

$$\sigma(v) = \left(\frac{1}{\lambda(-v)} - \lambda(v) \right) \frac{1}{\beta} \quad (13)$$

Hence,

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \lambda(v) \begin{pmatrix} 1 & \left(\frac{1}{\lambda(-v)\lambda(v)} - 1 \right) \frac{1}{\beta} \\ -\beta & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix} \quad (14)$$

Thus, the infinitesimal transformations are

$$cdt' = \lambda(v) \left(cdt + \left(\frac{1}{\lambda(-v)\lambda(v)} - 1 \right) \frac{1}{\beta} dx \right) \quad (15)$$

$$dx' = \lambda(v)(dx - \beta cdt) \quad (16)$$

From these equations, we can get that

$$\frac{dx'}{dt'} = \frac{\frac{dx}{dt} - \beta c}{1 + \left(\frac{1}{\lambda(-v)\lambda(v)} - 1 \right) \frac{1}{\beta c} \frac{dx}{dt}} \quad (17)$$

This is the velocity addition rule. We assume the speed of light c as the same in any system. Thus, considering $\frac{dx'}{dt'} = \frac{dx}{dt} = c$ we have that

$$c = \frac{c - \beta c}{1 + \left(\frac{1}{\lambda(-v)\lambda(v)} - 1 \right) \frac{1}{\beta c} c} \quad (18)$$

Therefore,

$$\frac{1}{\lambda(-v)\lambda(v)} = 1 - \beta^2 \quad (19)$$

We define

$$\gamma = \frac{1}{\sqrt{1 - \beta^2}} \quad (20)$$

and with this,

$$\lambda(-v)\lambda(v) = \gamma^2 \quad (21)$$

Using (21) in (14) we get

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \lambda(v) \begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix} \quad (22)$$

We can define $\varphi(v)\gamma = \lambda(v)$, thus

$$\varphi(-v)\varphi(v) = 1 \quad (23)$$

and

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \varphi(v)\gamma \begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix} \quad (24)$$

Thus, the transformations are:

$$ct' = \varphi(v)\gamma(ct - \beta x) \quad (25)$$

$$x' = \varphi(v)\gamma(x - \beta ct) \quad (26)$$

Under the assumption of isotropy $\varphi(v) = \varphi(-v)$. Using this in equation (23), $\varphi(v) = 1$ and we get the Lorentz transformations:

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \gamma \begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix} \quad (27)$$

This matricial equation contains the transformations

$$ct' = \gamma(ct - \beta x) \quad (28)$$

$$x' = \gamma(x - \beta ct) \quad (29)$$

Making a geometrical connection we can define $\tanh \theta = \beta = \frac{v}{c}$:

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} \cosh \theta & -\sinh \theta \\ -\sinh \theta & \cosh \theta \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix} \quad (30)$$

If we take the line element $ds^2 = c^2 dt^2 - dx^2$, in the primed frame:

$$c^2 dt'^2 - dx'^2 = \gamma^2 (cdt - \beta dx)^2 - \gamma^2 (dx - \beta cdt)^2$$

After some algebra:

$$c^2 dt'^2 - dx'^2 = c^2 dt^2 - dx^2 = ds^2 \quad (31)$$

Lorentz transformations left the line element invariant.

2 The anisotropic assumption

Following Bogoslovsky, we introduce a conformal factor $e^{-b\theta}$ in (30):

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} e^{-b\theta} \cosh \theta & -e^{-b\theta} \sinh \theta \\ -e^{-b\theta} \sinh \theta & e^{-b\theta} \cosh \theta \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix} \quad (32)$$

Since $\theta = \tanh^{-1} \beta = \frac{1}{2} \ln \left(\frac{1+\beta}{1-\beta} \right)$ we have $e^{-b\theta} = e^{-\frac{b}{2} \ln \left(\frac{1+\beta}{1-\beta} \right)} = \left(\frac{1-\beta}{1+\beta} \right)^{\frac{b}{2}}$ Therefore,

$$ct' = \left(\frac{1-\beta}{1+\beta} \right)^{\frac{b}{2}} \gamma(ct - \beta x) \quad (33)$$

$$x' = \left(\frac{1-\beta}{1+\beta} \right)^{\frac{b}{2}} \gamma(x - \beta ct) \quad (34)$$

The same result can be obtained if we don't assume isotropy in (24). Using $\tanh \theta = \beta = \frac{v}{c}$ we get

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \varphi(\theta) \begin{pmatrix} \cosh \theta & -\sinh \theta \\ -\sinh \theta & \cosh \theta \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix} \quad (35)$$

The product of two transformations is:

$$\varphi(\theta) \begin{pmatrix} \cosh \theta & -\sinh \theta \\ -\sinh \theta & \cosh \theta \end{pmatrix} \varphi(\psi) \begin{pmatrix} \cosh \psi & -\sinh \psi \\ -\sinh \psi & \cosh \psi \end{pmatrix} \quad (36)$$

$$= \varphi(\theta) \varphi(\psi) \begin{pmatrix} \cosh(\theta + \psi) & -\sinh(\theta + \psi) \\ -\sinh(\theta + \psi) & \cosh(\theta + \psi) \end{pmatrix} \quad (37)$$

To get a group the following should be satisfied:

$$\varphi(\theta) \varphi(\psi) = \varphi(\theta + \psi) \quad (38)$$

Dhasmana and Silagadze say that all the continuous solutions of (38) are of the form $\varphi(\theta) = e^{-b\theta}$, which is the Bogoslovsky term.

These transformations don't left invariant the standard line element

$$ds^2 = c^2 dt^2 - dx^2$$

We observe that

$$cdt'^2 - dx'^2 = \left(\frac{1-\beta}{1+\beta} \right)^b (cdt^2 - dx^2) = \left(\frac{1-\beta}{1+\beta} \right)^b ds^2 \quad (39)$$

However they left invariant

$$ds^2 = \left(\frac{(cdt - dx)^2}{cdt^2 - dx^2} \right)^b (cdt^2 - dx^2) \quad (40)$$

The new line element can be written as

$$ds^2 = (cdt - dx)^{2b} (c^2 dt^2 - dx^2)^{1-b} \quad (41)$$

invariant under

$$ct' = \left(\frac{1-\beta}{1+\beta} \right)^{\frac{b}{2}} \gamma(ct - \beta x) \quad (42)$$

$$x' = \left(\frac{1-\beta}{1+\beta} \right)^{\frac{b}{2}} \gamma(x - \beta ct) \quad (43)$$

References

- [1] G. Y. Bogoslovsky, “DISIM_b(2) Local Relativistic Symmetry and Finslerian Extension of the Theory of Relativity,” [arXiv:2002.02843 [gr-qc]].
- [2] S. Dhasmana and Z. K. Silagadze, “Finsler spacetime in light of Segal’s principle,” Mod. Phys. Lett. A **35** (2019) no.06, 2050019 doi:10.1142/S0217732320500194 [arXiv:1911.04233 [gr-qc]].