

Here, we will analyse that a fermion in a plane wave background, typically used to model lasers, presents similar equations compared to Very Special Relativity (VSR), which is a model to describe the existence of neutrino mass as we assume that the nature symmetry is not Lorentz, but a subgroup of it, usually SIM(2) [1]. This similitude has been described for example in a work of A. Ilderton [2].

Let us start with the typical Dirac equation for an electron with mass M in an external electromagnetic field:

$$[(i\cancel{\partial} - e\cancel{A}) - M] \psi = 0 \quad (1)$$

and we will consider that the external field is a plane wave. Hence,

$$A_\mu = A_\mu(k \cdot x)$$

Acting with the operator $[(i\cancel{\partial} - e\cancel{A}) + M]$ in the left, using the anticommutation relation of the gamma matrices

$$\gamma^\mu \gamma^\nu = 2\eta^{\mu\nu} - \gamma^\nu \gamma^\mu$$

we have that

$$[-\partial^2 - ie\gamma^\mu \gamma^\nu (\partial_\mu A_\nu) - 2ieA \cdot \partial + e^2 A^2 - M^2] \psi = 0 \quad (2)$$

and since $A_\nu = A_\nu(k \cdot x)$, therefore, $\partial_\mu A_\nu = A'_\nu k_\mu$, where the prime stands for a derivative respect to A . Hence,

$$[-\partial^2 - ie\cancel{k}\cancel{A}' - 2ieA \cdot \partial + e^2 A^2 - M^2] \psi = 0 \quad (3)$$

Using as ansatz

$$\psi = e^{-ip \cdot x} g(k \cdot x) \quad (4)$$

and that $k^2 = 0$ since it is a electromagnetic wave-vector, and $p^2 = M^2$ for a fermion, we have

$$2ik \cdot p g'(k \cdot x) - ie\cancel{k}\cancel{A}' g(k \cdot x) - 2eA \cdot p g(k \cdot x) - 2ieA \cdot k g'(k \cdot x) + e^2 A^2 g(k \cdot x) = 0 \quad (5)$$

If A satisfies the Lorenz gauge $k \cdot A = 0$:

$$2ik \cdot p g'(k \cdot x) + [-ie\cancel{k}\cancel{A}' - 2eA \cdot p + e^2 A^2] g(k \cdot x) = 0 \quad (6)$$

Arranging:

$$g'(k \cdot x) = \frac{1}{2k \cdot p} [i(e^2 A^2 - 2eA \cdot p) + e\cancel{k}\cancel{A}'] g(k \cdot x) \quad (7)$$

Equation (7) is of the kind $g'(\varphi) = R(\varphi)g(\varphi)$ with solution

$$g(\varphi) = c_1 \exp \left[\int_0^\varphi d\bar{\varphi} R(\bar{\varphi}) \right] \quad (8)$$

(More precisely for our case: $g'(\varphi) = S'(\varphi)g(\varphi)$ and the solution here is $g(\varphi) = c_2 \exp[S(\varphi)]$). Using it, we have

$$g(k \cdot x) = C \exp \left[i \int_0^{k \cdot x} d\bar{\varphi} \left(\frac{e^2 A^2 - 2eA \cdot p}{2k \cdot p} \right) + e \frac{\not{k} \not{A}}{2k \cdot p} \right] \quad (9)$$

Notice that

$$\exp \left[e \frac{\not{k} \not{A}}{2k \cdot p} \right] = 1 + e \frac{\not{k} \not{A}}{2k \cdot p} + e^2 \frac{\not{k} \not{A} \not{k} \not{A}}{4(k \cdot p)^2} + \dots + \quad (10)$$

Using the anticommutation relation of the gamma matrices, that $k \cdot A = 0$ since we assume that the field satisfies the Lorenz gauge, and $\not{k} \not{k} = k^2 = 0$ we have that the higher powers vanishes and we are left with

$$\exp \left[e \frac{\not{k} \not{A}}{2k \cdot p} \right] = 1 + e \frac{\not{k} \not{A}}{2k \cdot p} \quad (11)$$

Hence,

$$g(k \cdot x) = C \left(1 + e \frac{\not{k} \not{A}}{2k \cdot p} \right) \exp \left[i \int_0^{k \cdot x} d\bar{\varphi} \left(\frac{e^2 A^2 - 2eA \cdot p}{2k \cdot p} \right) \right] \quad (12)$$

It allows us to define the state ψ as

$$\psi = \exp[-iS_p(x)] \Gamma_p(k \cdot x) u_{p,s} \quad (13)$$

where

$$S_p(x) = p \cdot x + \int_0^{k \cdot x} d\bar{\varphi} \left(\frac{2eA \cdot p - e^2 A^2}{2k \cdot p} \right) \quad (14)$$

$$\Gamma_p(k \cdot x) = \left(1 + e \frac{\not{k} \not{A}}{2k \cdot p} \right) \quad (15)$$

Considering the possibility to start with p with opposite sign in our ansatz, the most general way to write the state is

$$\psi = \exp[-iS_p(x)] \Gamma_p(k \cdot x) u_{p,s} + \exp[-iS_{-p}(x)] \Gamma_{-p}(k \cdot x) v_{p,s} \quad (16)$$

where we observe that if $A \rightarrow 0$: $\psi = e^{-ip \cdot x} u_{p,s} + e^{ip \cdot x} v_{p,s}$, which is the free solution. This solution has been known for long time, from 1935, due to Volkov [3].

We come back to the Dirac equation (1) and we use ψ explicitly as in equation (13) and after some algebra we have

$$[i\gamma^\mu \partial_\mu + \gamma^\mu \partial_\mu S_p(x) - e\gamma^\mu A_\mu - M][\Gamma_p(k \cdot x) u_{p,s}] = 0 \quad (17)$$

We use that

$$\frac{d}{dx} \int_{a(x)}^{b(x)} f(x, t) dt = f(x, b(x)) \frac{db}{dx} - f(x, a(x)) \frac{da}{dx} + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x, t) dt \quad (18)$$

to obtain

$$\partial_\mu S_p(x) = \partial_\mu \left(p \cdot x + \int_0^{k \cdot x} d\bar{\varphi} \left(\frac{2eA \cdot p - e^2 A^2}{2k \cdot p} \right) \right) = p_\mu + \left(\frac{2eA \cdot p - e^2 A^2}{2k \cdot p} \right) k_\mu \quad (19)$$

With this,

$$i \not{k} \Gamma'_p(k \cdot x) + (\not{q} - m) \Gamma_p(k \cdot x) = 0 \quad (20)$$

where we have defined the momentum

$$q_\mu = p_\mu - eA_\mu + \left(\frac{2eA \cdot p - e^2 A^2}{2k \cdot p} \right) k_\mu \quad (21)$$

Deriving Γ_p we have

$$\Gamma'_p(k \cdot x) = -e \frac{\not{k} A'_\nu(k \cdot x)}{2k \cdot p} \quad (22)$$

Therefore,

$$\not{k} \Gamma'_p(k \cdot x) = 0 \quad (23)$$

Hence,

$$(\not{q} - m) \Gamma_p(k \cdot x) = 0 \quad (24)$$

Now averaging the momentum q_μ :

$$\langle q_\mu \rangle = p_\mu - \frac{\langle e^2 A^2 \rangle}{2} \frac{k_\mu}{k \cdot p} \quad (25)$$

where we are considering that in average only the amplitude of A (i.e. A^2) is constant. Any other term containing A vanishes in average. Then, averaging ψ in (16), we get

$$\langle \psi \rangle = \langle \exp[-iS_p(x)] \Gamma_p(k \cdot x) \rangle u_{p,s} + \langle \exp[-iS_{-p}(x)] \Gamma_{-p}(k \cdot x) \rangle v_{p,s} \quad (26)$$

In parallel,

$$\langle \exp[-iS_p(x)] \Gamma_p(k \cdot x) \rangle = \exp \left[-i \left(p \cdot x - \int_0^{k \cdot x} d\bar{\varphi} \left(\frac{e^2 \langle A^2 \rangle}{2k \cdot p} \right) \right) \right] \quad (27)$$

Since we are considering $\langle A^2 \rangle$ constant, we take it out of the integral and

$$\int_0^{k \cdot x} d\bar{\varphi} \left(\frac{e^2 \langle A^2 \rangle}{2k \cdot p} \right) \approx \frac{e^2 \langle A^2 \rangle}{2k \cdot p} k \cdot x \quad (28)$$

Therefore,

$$\langle \exp[-iS_p(x)] \Gamma_p(k \cdot x) \rangle \approx \exp \left[-i \left(p - \frac{\langle e^2 A^2 \rangle}{2k \cdot p} k \right) \cdot x \right] = e^{-i \langle q \rangle \cdot x} \quad (29)$$

Hence,

$$\langle \psi \rangle = e^{-i\langle q \rangle \cdot x} u_{p,s} + e^{i\langle q \rangle \cdot x} v_{p,s} \quad (30)$$

With this, we average in equation (1) and we have

$$\left[\not{p} - \frac{\langle e^2 A^2 \rangle}{2} \frac{\not{k}}{k \cdot p} - M \right] \langle \psi \rangle = e \langle \not{A} \psi \rangle \quad (31)$$

After a long computation we can show that

$$e \langle \not{A} \psi \rangle = -e^2 \frac{\not{k} \langle A^2 \rangle}{2k \cdot p} e^{-i\langle q \rangle \cdot x} u_{p,s} \quad (32)$$

Hence,

$$[\not{p} - M] \langle \psi \rangle = 0 \quad (33)$$

Notice that this description is equivalent to start from an equation

$$\left[i\not{\partial} - i \frac{\langle e^2 A^2 \rangle}{2} \frac{\not{k}}{k \cdot \partial} - M \right] \langle \psi \rangle = 0 \quad (34)$$

for the average fermion field.

If we define $e \langle e^2 A^2 \rangle = -m^2$ we have

$$\left[i\not{\partial} + i \frac{m^2}{2} \frac{\not{k}}{k \cdot \partial} - M \right] \langle \psi \rangle = 0 \quad (35)$$

which corresponds to the fermionic VSR equation, as we can see as example in equation (7) of a previous work [4].

References

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